

The cluster value problem for Banach spaces.[‡]

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Abstract

The main result is that the cluster value problem in separable Banach spaces, for the Banach algebras A_u and H^∞ , can be reduced to the cluster value problem in those spaces which are ℓ_1 sums of a sequence of finite dimensional spaces.

1 Introduction

The cluster value problem was introduced in 1961 in [3] by a group of mathematicians who worked together under the alias I. J. Scharf. They compared the set of cluster values of a function f in an algebra H of bounded analytic functions on a domain U at a point $z \in \bar{U}$ with the set $\{\gamma(f) : \gamma \in \mathcal{M}_z\}$ where \mathcal{M}_z is the set of multiplicative linear functionals on the algebra H such that $\gamma(L) = L(z)$ for all linear functionals. They found these sets are the same when the domain U is the unit disk in the complex plane. It was soon understood that this cluster value theorem would be a consequence of a positive solution to the famous corona problem, which of course was solved positively in the unit disk of the complex plane by Carleson in 1962. In 1979 the cluster value problem was solved positively by McDonald in [7] for bounded analytic functions defined on a strongly pseudoconvex domain with smooth boundary in \mathbb{C}^n . But only in 2012 did Aron, Carando, Gamelin, Lasalle and Maestre

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consider the case of domains U in infinite dimensional Banach spaces. They solved the cluster value problem at the origin for bounded, uniformly continuous and analytic functions defined on the ball of a Banach space with a shrinking 1-unconditional basis in [2]. This class of spaces include, for example, the ℓ_p spaces for $1 < p < \infty$ as well as c_0 , but not, for example, the space ℓ_1 or ℓ_∞ or $L_p(0, 1)$, $1 \leq p \neq 2 \leq \infty$. In [5] we gave a positive answer to the cluster value problem for bounded analytic functions on the ball of the space of continuous functions on a dispersed compact Hausdorff space.

This paper focuses attention on the cluster value problem for the unit ball of the ℓ_1 sum of finite dimensional spaces. We prove that in order to solve the cluster value problem for the unit ball of all separable Banach spaces, it is enough to solve it for the ball of spaces that are an ℓ_1 sum of a sequence of finite dimensional spaces. That is, if the cluster value problem has a negative solution for some separable Banach space, then it has a negative answer for some space that is an ℓ_1 sum of finite dimensional spaces. Although we do not see a reduction of the cluster value problem for all Banach spaces to the case of separable Banach spaces, an obvious modification of the proof of Theorem 1 yields that if the cluster value problem has a positive solution for (uncountable) ℓ_1 sums of all collections of finite dimensional spaces, then there is a positive answer for all Banach spaces.

To be precise, we recall [5] that the cluster value problem is posed as follows: Given a Banach space X , an algebra H of bounded analytic functions over the ball B of X that contains X^* , and a point x^{**} in the weak* closure of $B_{X^{**}}$ (that is, in the closed unit ball of X^{**}), we define, for $f \in H$, the set $Cl_B(f, x^{**})$ to be all the limit values of $f(x_\alpha)$ over all nets (x_α) in B_X converging to x^{**} in the weak* topology, and we define $M_{x^{**}}$ to be the multiplicative linear functionals in the spectrum over H that, when restricted to X^* , coincide with x^{**} . Is it then true that $Cl_B(f, x^{**}) \supset M_{x^{**}}(f)$ for every $f \in H$? The reverse inclusion is always true as we may recall from the first section in [5].

The algebras of analytic functions we consider in this paper are $H^\infty(B)$, the set of all bounded analytic functions on the open unit ball $B = B_X$ of the Banach space X , and $A_u(B)$, the uniformly continuous functions that are in $H^\infty(B)$.

2 Reduction of the cluster value problem to ℓ_1 sums of finite dimensional spaces

We will see that the cluster value problem in Banach spaces can be reduced to the cluster value problem in those spaces that are ℓ_1 sums of finite dimensional spaces. For simplicity, we will only show that the cluster value problem for separable spaces can be reduced to the cluster value problem in spaces that are a countable ℓ_1 sum of finite dimensional spaces, and it will be clear that the nonseparable case reduces to uncountable ℓ_1 sums of finite dimensional spaces.

We will need the next two lemmas. The ideas in the following lemmas originated with C. Stegall [8] and were developed by the first author in [4].

Lemma 1. *Let Y be a separable Banach space and $Y_1 \subset Y_2 \subset Y_3 \subset \dots$ an increasing sequence of finite dimensional subspaces whose union is dense in Y . Set $X = (\sum Y_n)_1$. Then the isometric quotient map $Q : X \rightarrow Y$ defined by*

$$Q(z_n)_n := \sum_{n=1}^{\infty} z_n$$

induces an isometric algebra homomorphism $Q^\# : H(B_Y) \rightarrow H(B_X)$, where H denotes either the algebra A_u or the algebra H^∞ .

Proof. Note that for all $(z_n)_n \in X$,

$$\|Q(z_n)_n\| = \left\| \sum_{n=1}^{\infty} z_n \right\| \leq \sum_{n=1}^{\infty} \|z_n\| = \|(z_n)_n\|_1. \quad (1)$$

Let $\widetilde{Y}_n = \{(z_n)_n \in X : z_k = 0 \forall k \neq n\}$. Since $Q(B_{\widetilde{Y}_n}) = B_{Y_n}$ for all $n \in \mathbb{N}$, we now have that $Q(B_X)$ is dense in B_Y and hence Q is an isometric quotient map.

Then the function $Q^\# : H(B_Y) \rightarrow H(B_X)$ given by $Q^\#(f) = f \circ Q$ is an isometric homomorphism because $Q^\#$ is clearly linear and for all $f, g \in H(B_Y)$,

$$\|Q^\#(f)\| = \sup_{x \in B_X} |f \circ Q(x)| = \sup_{y \in B_Y} |f(y)| = \|f\|.$$

Moreover,

$$Q^\#(f \cdot g) = (f \cdot g) \circ Q = (f \circ Q) \cdot (g \circ Q) = Q^\#(f)Q^\#(g),$$

so $Q^\#$ is an algebra homomorphism. ■

Lemma 2. *Under the assumptions of Lemma 1, there is a norm one algebra homomorphism $T : H(B_X) \rightarrow H(B_Y)$ so that $T(X^*) \subset Y^*$ and $T \circ Q^\# = I_{H(B_Y)}$.*

Proof. The first part of the proof consists of constructing T and verifying that $TH(B_X) \subset H(B_Y)$.

For every $y \in (\cup Y_n)$ and $n \in \mathbb{N}$, let $S_n(y) = (z_i)_i \in X$ be given by

$$z_i = \begin{cases} y & \text{if } i = n \text{ and } y \in Y_n \\ 0 & \text{otherwise} \end{cases}$$

Let \mathcal{U} be a free ultrafilter on \mathbb{N} . For each $g \in H(B_X)$ set

$$Sg(y) = \lim_{n \in \mathcal{U}} g(S_n y) \text{ for every } y \in B_{(\cup Y_n)},$$

which is well defined because g is bounded. Next we prove that Sg is continuous:

Let $0 < r < 1$. Since $g \in H^\infty(B_X)$ then Schwarz' Lemma (Thm. 7.19, [6]) and the convexity of B_X imply that $g \in A_u(rB_X)$. Let $\epsilon > 0$. Since g is uniformly continuous on rB_X there exists $\delta > 0$ such that, if $a, b \in rB_X$ and $\|a - b\| < \delta$, then $\|g(a) - g(b)\| < \epsilon$. Thus, given $y_1, y_2 \in rB_{(\cup Y_n)}$ such that $\|y_1 - y_2\| < \delta$, we can find $N \in \mathbb{N}$ such that $y_1, y_2 \in Y_n \ \forall n \geq N$, and then $\|S_n(y_1) - S_n(y_2)\| = \|y_1 - y_2\| < \delta$ eventually for n , so

$$\|Sg(y_1) - Sg(y_2)\| = \lim_{n \in \mathcal{U}} \|g(S_n(y_1)) - g(S_n(y_2))\| \leq \epsilon.$$

Each $Sg : B_{(\cup Y_n)} \rightarrow \mathbb{C}$ is uniformly continuous on $rB_{(\cup Y_n)}$ for $0 < r < 1$, thus we can continuously extend each Sg to $Tg : B_Y \rightarrow \mathbb{C}$. Moreover, it is evident that Tg is uniformly continuous on B_X when $g \in A_u(B_X)$. It is left to show that each Tg is analytic by checking that every Tg is analytic in each complex line (Thm. 8.7, [6]). We do this in two parts.

Step 1 Let us check that each Sg is analytic on complex lines:

Let $y^1 \in B_{(\cup Y_n)}$ and $y^2 \neq 0 \in (\cup Y_n)$. Since $B_{(\cup Y_n)}$ is open we can find $R > 0$ such that, if $\|y - y^1\| < R$, then $y \in B_{(\cup Y_n)}$. Choose $r > 0$ such that $r\|y^2\| < R$. Thus, if $|\lambda| \leq r$ we have that $\lambda \in \Lambda = \{\zeta \in \mathbb{C} : y^1 + \zeta y^2 \in B_{(\cup Y_n)}\}$.

For each $n \in \mathbb{N}$, let $u_n = S_n(y^1)$ and $w_n = S_n(y^2)$.

Using the notation in Remark 5.2 in [6], we claim that

$$Sg(y^1 + \lambda y^2) = \sum_{m=0}^{\infty} (\lim_{n \in \mathcal{U}} P^m g(u_n)(w_n)) \lambda^m,$$

uniformly on λ , for $|\lambda| \leq r$. Let us start by showing that for each $m \in \mathbb{N}$ and $\lambda \in \bar{\Delta}(0, r)$, $\lim_{n \in \mathcal{U}} P^m g(u_n)(\lambda w_n)$ exists.

We can find $s > 1$ such that also $sr\|y^2\| < R$. Then, when $|t| \leq s$ and $|\lambda| \leq r$, we have that $y^1 + t\lambda y^2 \in B_{(\cup Y_n)}$, because

$$\|(y^1 + t\lambda y^2) - y^1\| = |t|\lambda\|y^2\| \leq sr\|y^2\| < R,$$

so $u_n + t\lambda w_n \in B_X$ eventually for n , and from Cauchy's Inequality (Cor. 7.4, [6]), for each $m \in \mathbb{N}$

$$\|P^m g(u_n)(\lambda w_n)\| \leq \frac{1}{s^m} \|g\|,$$

eventually for n , and then $\lim_{n \in \mathcal{U}} P^m g(u_n)(\lambda w_n)$ exists.

Moreover, given $M \in \mathbb{N}$ and λ such that $|\lambda| \leq r$,

$$\begin{aligned} & \|Sg(y^1 + \lambda y^2) - \sum_{m=0}^M \lim_{n \in \mathcal{U}} P^m g(u_n)(w_n) \lambda^m\| \\ &= \|\lim_{n \in \mathcal{U}} (g(u_n + \lambda w_n) - \sum_{m=0}^M P^m g(u_n)(\lambda w_n))\| \\ &= \|\lim_{n \in \mathcal{U}} \sum_{m=M+1}^{\infty} P^m g(u_n)(\lambda w_n)\| \\ &\leq \sum_{m=M+1}^{\infty} \frac{1}{s^m} \|g\| = \frac{\|g\|}{s-1} \frac{1}{s^M}, \end{aligned}$$

which goes to zero as $M \rightarrow \infty$.

Thus Sg is analytic on complex lines.

Step 2 The following general lemma should be known, but we could not find a reference.

Lemma 3. *If $\phi : B_Y \rightarrow \mathbb{C}$ is bounded and uniformly continuous on sB_Y for each $0 < s < 1$, and there is a dense subspace Z of Y such that $\phi|_{B_Z}$ is analytic on complex lines, then ϕ is analytic on complex lines in B_Y (and hence analytic).*

Proof. Let $y^1 \in B_Y$ and $y^2 \neq 0 \in Y$. Let $s \in (\|y^1\|, 1)$. Since sB_Y is open and contains y^1 , we can find $R > 0$ such that, if $\|y - y^1\| < R$ then $y \in sB_Y$. Choose $r > 0$ such that $r\|y^2\| \leq R$. Thus, if $|\lambda| < r$ we have that $\lambda \in \Lambda = \{\zeta \in \mathbb{C} : y^1 + \zeta y^2 \in sB_Y\}$.

Let $f : \lambda \rightarrow \phi(y^1 + \lambda y^2)$, a function defined for $|\lambda| < r$. We want to show that f is analytic.

Let $\{y_k^1\}_k \subset B_Z$ and $\{y_k^2\}_k \subset Z$ be sequences such that $\|y_k^1 - y^1\| \leq \frac{1}{2^k}$ and $\|y_k^2 - y^2\| \leq \frac{1}{2^k}$. Choose $K_1 \in \mathbb{N}$ such that $\frac{1+r}{2^{K_1}} \leq R - r\|y^2\|$. Then for $k \geq K_1$ and $|\lambda| < r$ we have that

$$\begin{aligned} \| (y_k^1 + \lambda y_k^2) - y^1 \| &\leq \| y_k^1 - y^1 \| + |\lambda| \| y_k^2 - y^2 \| + |\lambda| \| y^2 \| \\ &< \frac{1+r}{2^{K_1}} + r \| y^2 \| \\ &\leq R, \end{aligned}$$

so $y_k^1 + \lambda y_k^2 \in sB_Z$.

For each $k \geq K_1$, let $f_k : \lambda \rightarrow \phi(y_k^1 + \lambda y_k^2)$, which is an analytic function for $|\lambda| < r$ by assumption.

Since ϕ is bounded, clearly $\{f_k\}_{k \geq K_1}$ is uniformly bounded. Let us now show that $\{f_k\}_{k \geq K_1}$ converges uniformly to f . Let $\epsilon > 0$. Since ϕ is uniformly continuous on sB_Z , we can find $\delta > 0$ such that,

$$a, b \in sB_Z, \quad \|a - b\| < \delta \implies \|\phi(a) - \phi(b)\| < \epsilon.$$

Choose $K \geq K_1$ such that $\frac{1+r}{2^K} < \delta$. Then $\forall k \geq K$ and λ with $|\lambda| < r$,

$$\begin{aligned}
\|(y_k^1 + \lambda y_k^2) - (y^1 + \lambda y^2)\| &\leq \|y_k^1 - y^1\| + |\lambda| \|y_k^2 - y^2\| \\
&\leq \frac{1+r}{2^k} \\
&< \delta,
\end{aligned}$$

so $\|f_k(\lambda) - f(\lambda)\| = \|\phi(y_k^1 + \lambda y_k^2) - \phi(y^1 + \lambda y^2)\| < \epsilon$.

Then, by the lemma on p. 226 in [1], f is analytic. ■

From the previous two steps, we obtain that T is a well defined mapping from $H(B_X)$ into $H(B_Y)$. Now, given $x^* \in X^*$, $y^1, y^2 \in B_Y$ and $\lambda \in \mathbb{C}$ such that $y^1 + \lambda y^2 \in B_Y$, we can find $\{y_k^1\}_k \subset B_{(\cup Y_n)}$ converging to y^1 and $\{y_k^2\}_k \subset B_{(\cup Y_n)}$ converging to y^2 , and then

$$\begin{aligned}
Tx^*(y^1 + \lambda y^2) &= \lim_{k \rightarrow \infty} \lim_{n \in \mathcal{U}} x^*(S_n(y_k^1 + \lambda y_k^2)) \\
&= \lim_{k \rightarrow \infty} \lim_{n \in \mathcal{U}} (x^*(S_n(y_k^1)) + \lambda x^*(S_n(y_k^2))) \\
&= Tx^*(y^1) + \lambda Tx^*(y^2),
\end{aligned}$$

i.e. $Tx^* \in Y^*$. This shows that $TB_{X^*} = B_{Y^*}$.

Moreover, for every $f \in H(B_Y)$ and $y \in B_Y$, we can find $\{y_k\}_k \subset B_{(\cup Y_n)}$ converging to y , and thus

$$\begin{aligned}
T \circ Q^\#(f)(y) &= T(f \circ Q)(y) \\
&= \lim_{k \rightarrow \infty} \lim_{n \in \mathcal{U}} f \circ Q(S_n(y_k)) \\
&= \lim_{k \rightarrow \infty} f(y_k) \\
&= f(y),
\end{aligned}$$

so $T \circ Q^\# = I_{H(B_Y)}$.

Also, T is a homomorphism because T is clearly linear and for all $f, g \in H(B_X)$, $y \in B_Y$, we can find $\{y_k\}_k \subset B_{(\cup Y_n)}$ converging to y , so

$$\begin{aligned}
T(f \cdot g)(y) &= \lim_{k \rightarrow \infty} \lim_{n \in \mathcal{U}} f \cdot g(S_n(y_k)) \\
&= \lim_{k \rightarrow \infty} \lim_{n \in \mathcal{U}} f(S_n(y_k)) \cdot g(S_n(y_k)) \\
&= Tf(y) \cdot Tg(y) \\
&= (Tf \cdot Tg)(y).
\end{aligned}$$

Finally, for every $f \in H(B_X)$,

$$\|Tf\| = \sup_{y \in B_Y} |Tf(y)| = \sup_{y \in B(\cup Y_n)} |\lim_{n \in \mathcal{U}} f(S_n(y))| \leq \sup_{x \in B_X} |f(x)| = \|f\|,$$

and $\|T\| = \|T\| \|Q^\#\| \geq \|T \circ Q^\#\| = 1$. So $\|T\| = 1$. ■

Theorem 1. *Let Y be a separable Banach space and $Y_1 \subset Y_2 \subset Y_3 \subset \dots$ an increasing sequence of finite dimensional subspaces whose union is dense in Y . Set $X = (\sum Y_n)_1$. Let H denote either the algebra A_u or the algebra H^∞ . If $H(B_X)$ satisfies the cluster value theorem at every $x^{**} \in \overline{B_X}^{**}$ then $H(B_Y)$ satisfies the cluster value theorem at every $y^{**} \in \overline{B_Y}^{**}$.*

Proof. We know that $\hat{f}(M_{x^{**}}(B_X)) \subset Cl_{B_X}(f, x^{**})$, for all $f \in H(B_X)$ and $x^{**} \in \overline{B_X}^{**}$. Let us show that $\hat{g}(M_{y^{**}}(B_Y)) \subset Cl_{B_Y}(g, y^{**})$ for all $g \in H(B_Y)$ and $y^{**} \in \overline{B_Y}^{**}$.

Let $y^{**} \in \overline{B_Y}^{**}$, $\tau \in M_{y^{**}}(B_Y)$ and $g \in H(B_Y)$. Let T be the algebra homomorphism from $H(B_X)$ to $H(B_Y)$ constructed in Lemma 2. Then $x^{**} = y^{**} \circ T \in \overline{B_X}^{**}$, $Q^\#(g) \in H(B_X)$ and defining $\tilde{\tau} = \tau \circ T$, we see that $\tilde{\tau} \in M_{x^{**}}(B_X)$ because for all $x^* \in X^*$,

$$\tilde{\tau}(x^*) = \tau(Tx^*) = \langle y^{**}, Tx^* \rangle = \langle x^{**}, x^* \rangle.$$

Moreover,

$$\widehat{Q^\#(g)}(\tilde{\tau}) = \tilde{\tau}(Q^\#(g)) = \tau(T \circ Q^\#(g)) = \tau(g) = \hat{g}(\tau)$$

and

$$Cl_{B_X}(Q^\#(g), x^{**}) = Cl_{B_X}(g \circ Q, x^{**}) \subset Cl_{B_Y}(g, Q^{**} x^{**}) = Cl_{B_Y}(g, y^{**}),$$

so the theorem is established. ■

Remark 1. A very special case of Theorem 1 is that if ℓ_1 satisfies the cluster value theorem, then so does L_1 . We do not know for $1 < p \neq 2 < \infty$ whether the cluster value theorem for ℓ_p implies the cluster value theorem for L_p . Incidentally, in [2] it was proved that ℓ_p for p in this range satisfies the cluster value theorem at 0, but it is open whether L_p satisfies the cluster value theorem at any point of B_{L_p} .

Remark 2. The analogue of Theorem 1 for non separable spaces, which is proved by a non essential modification of the proof of Theorem 1, can be stated as follows. *Let Y be a Banach space and $(Y_\alpha)_{\alpha \in A}$ a family of finite dimensional subspaces of Y that is directed by inclusion and whose union is dense in Y . If $(\sum_{\alpha \in A} Y_\alpha)_1$ satisfies the cluster value theorem, then so does Y .*

Remark 3. There is a slight strengthening of Theorem 1. *Let Y , $(Y_n)_n$, and H be as in the statement of Theorem 1 and suppose that $(X_n)_n$ is a sequence so that X_n is $1 + \epsilon_n$ -isomorphic to Y_n and $\epsilon_n \rightarrow 0$. If $(\sum_n X_n)_1$ satisfies the cluster value theorem for the algebra H , then so does Y .* Now let (Z_n) be a sequence of finite dimensional spaces so that for every finite dimensional space Z and every $\epsilon > 0$, the space Z is $1 + \epsilon$ -isomorphic to one (and hence infinitely many) of the spaces Z_n . Set $C_1 = (\sum_n Z_n)_1$. As an immediate consequence of this slight improvement of Theorem 1 we get *If C_1 satisfies the cluster value theorem for H , then so does every separable Banach space.*

The proof of the improved Theorem 1 is essentially the same as the proof of the theorem itself. One just needs to define in Lemma 1 the mapping Q so that the conclusion of Lemma 1 remains true: For each n take an isomorphism $J_n : X_n \rightarrow Y_n$ so that for $x \in X_n$ the inequality $(1 + \epsilon_n)^{-1} \|x\| \leq \|J_n x\| \leq \|x\|$ is valid, and define $Q(x_n)_n = \sum_n J_n x_n$ for $(x_n)_n$ in $(\sum_n X_n)_1$.

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